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On the Number of Invariant Measures for Higher-Dimensional Chaotic Transformations

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Let S be a bounded region in R^N and let $\mathscr{P} = \{S_i\}_{i=1}^m$ be a partition of S into a finite number of closed subsets having piecewise C^2 boundaries of finite $(N-1)$-dimensional measure. Let $\tau: S \to S$ be piecewise C^2 on $\mathscr{P}$ and expanding in the sense that there exists $0 < \sigma < 1$ such that for any $i = 1, 2, ..., m$, $\|DT_i^{-1}\| < \sigma$, where DT_i^{-1} is the derivative matrix of T_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. We prove that for some classes of such mappings, for example, Jabłonski transformations or convexity-preserving transformations, the number of crossing points constitutes a bound for the number of ergodic absolutely continuous τ-invariant measures. We give examples showing that in general the simple bound of one-dimensional dynamics cannot be generalized to higher dimensions. In fact, we show that it is possible to construct piecewise expanding C^2 transformations on a fixed partition with a finite number of elements but which have an arbitrarily large number of ergodic, absolutely continuous invariant measures.

KEY WORDS: Piecewise C^2 transformation on R^N; bound for number of ergodic, absolutely continuous invariant measures, crossing points of partition.

1. INTRODUCTION

Let S be a bounded region in R^N and let $\mathscr{P} = \{S_i\}_{i=1}^m$ be a partition of S into a finite number of closed subsets having piecewise C^2 boundaries of finite $(N-1)$-dimensional measure. Let $\tau: S \to S$ be piecewise C^2 on $\mathscr{P}$ and expanding in the sense that there exists $0 < \sigma < 1$ such that for any $i = 1, 2, ..., m$, $\|DT_i^{-1}\| < \sigma$, where DT_i^{-1} is the derivative matrix of T_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. Then, under general conditions,[10] it can be shown that τ has an absolutely continuous invariant measure (acim). The result in ref. 1 is a generalization of results proved in refs. 6, 8, and 17.

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In this note, we investigate the problem of bounding the number of acim's for N-dimensional transformations. In general, "dynamical systems... have a large set of invariant measure[15]." For example, in point transformation models for cellular automata, it is possible to have many acim's.[16] Related to the problem of the number of acim's is the question of the support of invariant densities. Knowing that the support has interior is important in our approach.

For one-dimensional transformations $\tau: I \to I$, $I = [0, 1]$, it is well known that the number of discontinuities of $\tau'(x)$ provides an upper bound for the number of independent acim's.[1] This result has been improved in refs. 2–5. The key to all these bounds lies in the fact that invariant densities for piecewise C^2 expanding transformations are of bounded variation. In one dimension, a density of bounded variation is bounded and it can be proved that its support consists of a union of closed intervals. A simple argument then shows that each point of discontinuity of τ' must lie in the largest closed interval—hence the upper bound on the number of acim's. In higher dimensions, however, the situation is not so simple. The much more complex geometrical setting and the complicated form of the definition of bounded variation[7] do not permit an easy generalization of the one-dimensional result. For example, in two dimensions, the variation in one direction is integrated in the other direction. It is this integration which allows a function of bounded variation in R^N to be unbounded and its support to be devoid of interior.

In this note, we obtain a general bound on the number of independent acim's for Jabłonski transformations which are sufficiently expansive. This is accomplished by generalizing to N dimensions a result of ref. 6, based on a lemma in ref. 18, which allows us to prove that the support of every invariant density is open modulo a set of λ-measure zero, where λ is an N-dimensional Lebesgue measure, without explicitly using the definition of bounded variation in R^N.

In Section 2 we establish the main results of this note and in Section 3 we present a number of higher-dimensional examples. In Section 4 we show by means of an example that it is possible to construct piecewise expanding C^2 transformations on a fixed partition with a finite number of elements but which have arbitrarily large number of ergodic, absolutely continuous invariant measures.

2. MAIN RESULT

Let S be a bounded region in R^N and let τ be a transformation from S into S. We assume that τ is piecewise C^2 and expanding, i.e.:

(a) There exists a partition $\mathscr{P} = \{S_i\}_{i=1}^m$ of S, where m is a positive integer, and each S_i is a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure.

(b) $\tau_i = \tau_{|S_i}$ is a C^2, 1–1 transformation from $\mathrm{int}(S_i)$ onto its image and can be extended as a C^2 transformation on S_i, $i = 1, 2,..., m$.

(c) There exists $0 < \sigma < 1$ such that for any $i = 1, 2,..., m$,

$$\|D\tau_i^{-1}\| < \sigma \tag{1}$$

where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm.

We remark that condition (1) implies, for $\tau_1^{-1}(x)$, $\tau_1^{-1}(y)$ close enough,

$$\rho(\tau_1^{-1}(x), \tau_1^{-1}(y)) < \sigma\rho(x, y)$$

where $x, y \in R_1 = \tau(\mathrm{int}(S_1))$ and ρ is the Euclidean metric in R^N.

By λ we denote the Lebesgue measure on S, and by $J(\tau)$ the absolute value of the Jacobian of τ. $J(\tau, x)$ is the value of $J(\tau)$ at x.

The main tool used in the proofs of the existence of an acim for piecewise C^2 and expanding transformations is the multidimensional notion of variation defined using derivatives in the distributional sense[7]:

$$V(f) = \int_{R^N} \|Df\| = \sup\left\{\int_{R^N} f \operatorname{div}(g)\, d\lambda\colon g = (g_1,..., g_N) \in C_0^1(R^N, R^N) \text{ and } |g(x)| \leqslant 1,\ x \in R^N\right\}$$

where $f \in L_1(R^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(R^N, R^N)$ is the space of continuously differentiable functions from R^N into R^N having compact support.

In the sequel we shall consider the Banach space (ref. 7, Remark 1.12)

$$BV(S) = \{f \in L_1(S)\colon V(f) < +\infty\}$$

with the norm $\|f\|_{BV} = \|f\|_{L_1} + V(f)$.

By a regular cone in R^N we mean a cone whose base is an $(N-1)$-dimensional disk B and such that the central ray L joining the vertex to the center of the disk B is perpendicular to the disk. We define the angle subtended at the vertex of a regular cone to be angle between L and any line joining the vertex to a point on the boundary of B.

For any $S_i \in \mathscr{P}$, $i = 1, 2,..., m$, we define $a(S_i)$ as follows: at any singular point $x \in \partial S_i$ we construct the largest possible regular cone having its vertex at x and which lies completely in S_i. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_i) = \min_x \theta(x)$$

where x is a singular point in ∂S_i. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$a(S_i) = |\cos \alpha(S_i)|$$

We define

$$a = \min\{a(S_i): i = 1,..., m\}$$

We now state the main result of ref. 10, which will be needed in the sequel.

Theorem 1. Let $\tau: S \to S$, $S \subset R^N$, be a piecewise C^2, expanding transformation. If $\sigma(1 + 1/a) < 1$, then τ admits an absolutely continuous invariant measure with density $f \in BV(S)$.

The bounded variation inequalities proved in ref. 10 allow us to invoke the Ionescu Tulcea and Marinescu theorem,[20] which in particular says that the asymptotic σ-algebra $\mathfrak{A}_\infty(\tau)$ of τ is finite (λ – a.e.) or, in other words, that the number of ergodic acim's is finite. We recall that $\mathfrak{A}_\infty(\tau) = \bigcap_{n \geq 0} \tau^{-n}(\mathfrak{B})$, where $\mathfrak{B}$ is the Borel σ-algebra of subsets of S.

Theorem 1 is a generalization of results in refs. 6, 8, and 17. In refs. 8 and 17 the partitions are assumed to be rectangular and in ref. 17 the transformations are such that the ith component is a function only of the ith variable. The results in ref. 6 apply only to dimension 2, but in the special case when the transformation is piecewise analytic, $\sigma < 1$ is sufficient to establish the existence of an acim. For all the foregoing results, the densities are functions of bounded variation and the Ionescu Tulcea and Marinescu theorem applies.

Below we shall prove that the support of the density of any acim is open λ-a.e. This will be used to provide an explicit bound for the number of ergodic acim's.

We will now prove a result which is a direct extension of Theorem 2 in ref. 6 to N dimensions. First we state Lemma 1, which is an extension of Lemma 7 from ref. 6, and which in turn is based on ref. 18. For completeness, and since the relevant results of ref. 6 have not been published, we include the proof, which goes through with only minor changes.

Lemma 1. Let τ: $S \to S$, $S \subset R^N$, be a piecewise C^2 and expanding transformation and let f be the density of the acim μ, where $f \in L_p = L_p(S, \lambda)$ and $p > 1$. Let $q = p/(p-1)$. Let $\mathcal{Q} = \{Q_1,..., Q_M\}$ be a partition of S into bounded closed domains having piecewise C^2-boundaries of finite $(N-1)$-dimensional measure. We define

$$\mathscr{Q}^{(n)} = \bigvee_{k=0}^{n} \tau^{-k}(\mathscr{Q}) = \{Q_{i_0} \cap \tau^{-1}(Q_{i_1}) \cap \cdots \cap \tau^{-n}(Q_{i_n}) \colon Q_{i_j} \in \mathscr{Q}\}$$

Then:

(a) There exists $\alpha > 1$ such that

$$\operatorname{diam}(Q) \leqslant \operatorname{diam}(S)\, \alpha^{-n}$$

for any $Q \in \mathscr{Q}^{(n)}$.

(b) For any $\beta > 0$, there exists a positive integer $K(\beta)$ such that for any $k \geqslant K(\beta)$ and any $n \geqslant 0$ we can find a collection of sets $\mathscr{B}_{n,k} \subseteq \mathscr{Q}^{(n+k)}$ satisfying the following conditions:

(i) $\tau^n(B) \in \mathscr{Q}^{(k)}$, for any $B \in \mathscr{B}_{n,k}$.

(ii) $$\left| \frac{\lambda(\tilde{B})}{\lambda(B)} - \frac{\lambda(\tau^n(\tilde{B}))}{\lambda(\tau^n(B))} \right| \leqslant \beta \frac{\lambda(\tilde{B})}{\lambda(B)}$$

for any $B \in \mathscr{B}_{n,k}$ and any measurable $\tilde{B} \subset B$.

(iii) $\mu(S \setminus \bigcup \mathscr{B}_{n,k}) \leqslant \beta$.

Remark 1. If $f \in BV(S)$, then $f \in L_p$ with $p = N/(N-1)$ (see ref. 7).

Proof. We can assume that the partition $\mathscr{Q}$ is finer then the defining partition $\mathscr{P}$. Then (a) is obviously satisfied for any $1 < \alpha < 1/\sigma$.

Now we will prove (b). Let $k > 0$ and $n \geqslant 0$. For $Q \in \mathscr{Q}$ and $0 \leqslant j \leqslant n$, we define

$$S(n, k, j, Q) = \{x \in S \colon \operatorname{dist}(\tau^{k+j}(x), \partial(\tau(Q))) \leqslant \operatorname{diam}(S)\, \alpha^{-(k+n-j)}\}$$

and

$$S(n, k) = \bigcup_{Q \in \mathscr{Q}} \bigcup_{j=0}^{n} S(n, k, j, Q)$$

Then we have

$$\begin{aligned}
\mu(S(n,k)) &\leqslant \sum_{Q \in \mathscr{Q}} \sum_{j=0}^{n} \mu(S(n, k, j, Q)) \\
&= \sum_{Q \in \mathscr{Q}} \sum_{j=0}^{n} \int_S f \chi_{S(n,k,j,Q)}\, d\lambda \\
&\leqslant \sum_{Q \in \mathscr{Q}} \sum_{j=0}^{n} \|f\|_{L_p} (\lambda(S(n, k, j, Q)))^{1/q} \\
&\leqslant \|f\|_{L_p} \sum_{Q \in \mathscr{Q}} \sum_{j=0}^{n} (C_Q \operatorname{diam}(S)\, \alpha^{-(k+n-j)})^{1/q} \\
&\leqslant \|f\|_{L_p} C \alpha^{-k/q} (\alpha^{1/q}/(\alpha^{1/q} - 1))
\end{aligned}$$

where C_Q are constants dependent on $\partial(\tau(Q))$, and $C = \sum_{Q \in \mathscr{Q}} (C_Q \operatorname{diam}(S))^{1/q}$.

For any $\beta > 0$, we can find $K(\beta)$ such that for $k > K(\beta)$, $\mu(S(n, k)) < \beta$. For $k > K(\beta)$ and $n \geqslant 0$, we define

$$\mathscr{B}_{n,k} = \{B \in \mathscr{Q}^{(n+k)}: \text{for all } 1 \leqslant j \leqslant n,\ \tau^j(B) \in \mathscr{Q}^{(n+k-j)}\}$$

If $Q \in \mathscr{Q}^{(n+k)} \backslash \mathscr{B}_{n,k}$, then there exists a smallest j, $1 \leqslant j \leqslant n$, such that $\tau^j(Q) \notin \mathscr{Q}^{(n+k-j)}$. Then $\tau^{j-1}(Q) \in \mathscr{Q}^{(n+k-j+1)}$, which implies that there exist $Q_1 \in \mathscr{Q}$ and $Q_2 \in \mathscr{Q}^{(n+k-j)}$ such that $\tau^{j-1}(Q) = Q_1 \cap \tau^{-1}(Q_2)$ or $\tau^j(Q) = \tau(Q_1) \cap Q_2$ and $\tau^j(Q) \neq Q_2$. Since $\operatorname{diam}(\tau^j Q) \leqslant (\operatorname{diam}(S)\, \alpha^{-(k+n-j)})$, $Q \subseteq S(n, k)$.

We have proved that $\mu(S \backslash \bigcup \mathscr{B}_{n,k}) \leqslant \mu(S(n, k)) < \beta$ for $k \geqslant K(\beta)$, $n \geqslant 0$. Thus, (i) and (iii) are proved. We will now prove (ii).

Fix $B \in \mathscr{B}_{n,k}$, $x_0 \in B$, and $\tilde{B} \subset B$, and $0 \leqslant j \leqslant n-1$,

$$\begin{aligned}
\lambda(\tau^{j+1}\tilde{B}) &= \int_{\tau^j(\tilde{B})} J(\tau, x)\, d\lambda(x) \\
&= J(\tau, \tau^j(x_0)) \int_{\tau^j(\tilde{B})} \frac{J(\tau, x)}{J(\tau, \tau^j(x_0))}\, d\lambda(x)
\end{aligned}$$

On the other hand, we have for $x, y \in \tau^j B$,

$$\begin{aligned}
\left|\log\left(\frac{J(\tau, x)}{J(\tau, y)}\right)\right| &\leqslant \sup_{z \in \{x, y\}} \frac{1}{J(\tau, z)} |J(\tau, x) - J(\tau, y)| \\
&\leqslant \sigma^N L\, \|x - y\| \leqslant \sigma^N \operatorname{diam}(S)\, \alpha^{-(n+k-j)}
\end{aligned}$$

where L is the Lipschitz constant for $J(\tau)$. Hence, we have

$$\begin{aligned}
&\left|\log\left(\frac{\lambda(\tau^{j+1}\tilde{B})}{\lambda(\tau^{j+1}B)} \Big/ \frac{\lambda(\tau^j\tilde{B})}{\lambda(\tau^j B)}\right)\right| \\
&\quad = \left|\log\left(\frac{\int_{\tau^j(\tilde{B})} [J(\tau, x)/J(\tau, \tau^j(x_0))]\, d\lambda(x)}{\int_{\tau^j(B)} [J(\tau, x)/J(\tau, \tau^j(x_0)]\, d\lambda(x)} \Big/ \frac{\lambda(\tau^j\tilde{B})}{\lambda(\tau^j B)}\right)\right| \\
&\quad = \left|\log\left(\frac{J(\tau, x_1)}{J(\tau, x_2)}\right)\right| \leqslant C_1 \alpha^{-(n+k-j)}
\end{aligned}$$

By induction,

$$\left|\log\left(\frac{\lambda(\tau^n\tilde{B})}{\lambda(\tau^n B)} \Big/ \frac{\lambda(\tilde{B})}{\lambda(B)}\right)\right| \leqslant C_2 \alpha^{-k}$$

Hence,

$$\left|\frac{\lambda(\tilde{B})}{\lambda(B)}-\frac{\lambda(\tau^n(\tilde{B}))}{\lambda(\tau^n(B))}\right| \leqslant C_2\frac{\lambda(\tilde{B})}{\lambda(B)}\alpha^{-k}$$

where C_1 and C_2 are constants independent of k, n, B, and $\tilde{B}$. Since we can choose $K(\beta)$ such that $C_2\alpha^{-K(\beta)}<\beta$, (ii) is proved. ∎

Theorem 2. Let τ satisfy the assumptions of Lemma 1 and let the asymptotic σ-algebra $\mathfrak{A}_\infty(\tau)$ of τ be finite λ-a.e. Then:

(a) The atoms of $\mathfrak{A}_\infty(\tau)$ of τ are open sets λ-a.e.

(b) For any $A\in\mathfrak{A}_\infty(\tau)$, there exist $p\in\mathbb{N}$ and atoms $A_0,..., A_{p-1}\in\mathfrak{A}_\infty(\tau)$, pairwise disjoint, such that $A=A_0$, $\tau(A_{i-1})\subseteq A_i$, $\lambda-$a.e. $(i=1, 2,..., p-1)$, and $\tau(A_{p-1})\subseteq A$, $\lambda-$a.e.

Proof. (The following proof is almost directly from ref. 6. Since this result has not been published, we repeat the proof here.)

First we prove (b): Let A be an atom of $\mathfrak{A}_\infty(\tau)$, $A=\tau^{-n}(B_n)$, $B_n\in\mathfrak{B}$, $n=1, 2,...$, $\mu(A)>0$. The set $\tilde{A}=\bigcap_{k\geqslant 1}\bigcup_{n\geqslant k}\tau^{-n}(B_{n+1})$ is also an atom of $\mathfrak{A}_\infty(\tau)$ and $\tau(A)\subseteq\tilde{A}$. Since the number of atoms of $\mathfrak{A}_\infty(\tau)$ is finite (λ-a.e.), there exists a positive integer p such that $\tau^p(A)\subseteq A$, λ-a.e.

Now we prove (a): Let A be an atom of $\mathfrak{A}_\infty(\tau)$ and $\tau^p(A)\subseteq A$, λ-a.e. For $\beta=\mu(A)/2>0$, let us choose $k=K(\beta)$ and $\mathfrak{B}_{n,k}$ satisfying Lemma 1. Let $R_n=\bigcup\mathfrak{B}_{n,k}$ and $R^*=\bigcap_{l\geqslant 1}\bigcup_{n\geqslant l}R_{np}$. By (iii) of Lemma 1, we have

$$\mu(S\backslash R^*)\leqslant\beta=\mu(A)/2 \tag{2}$$

σ-algebras $\mathfrak{A}_{n+k}$, generated by partitions $\mathscr{Q}^{(n+k)}$, $n=0, 1,...$, form an increasing sequence converging to the Borel σ-algebra $\mathfrak{B}$. Thus,

$$\lim_{n\to\infty}E(\chi_{S\backslash A}\,|\,\mathfrak{A}_{n+k})(x)=\chi_{S\backslash A}(x), \qquad \lambda\text{-a.e.}$$

By (2), there exists a point $x_0\in R^*\cap A$ such that

$$\lim_{n\to\infty}E(\chi_{S\backslash A}\,|\,\mathfrak{A}_{n+k})(x_0)=0 \tag{3}$$

Since $x_0\in R^*$, there exists an increasing sequence of positive integers n_i and a sequence of sets $B_{n_i}\in\mathfrak{B}_{n_ip,k}$, $i=1, 2,...$, such that $x_0\in B_{n_i}$. By (3), we have

$$\lim_{i\to\infty}\frac{\lambda(B_{n_i}\backslash A)}{\lambda(B_{n_i})}=0$$

and by (ii) of Lemma 1

$$\lim_{i\to\infty} \frac{\lambda(\tau^{n_i p}(B_{n_i}\setminus A))}{\lambda(\tau^{n_i p}(B_{n_i}))} = 0$$

By (i) of Lemma 1

$$\tau^{n_i p}(B_{n_i}) \in \mathcal{Q}^{(k)}$$

for any $i = 1, 2, \ldots$, and since $\mathcal{Q}^{(k)}$ is finite, there exists a set $\tilde{Q} \in \mathcal{Q}^{(k)}$ such that

$$\tilde{Q} = \tau^{n_i p}(B_{n_i})$$

for natural numbers i from an infinite set $I \subset \mathbb{N}$. Thus,

$$\lambda(\tilde{Q}\setminus A) \leqslant \lim_{\substack{i\to\infty\\ i\in I}} \lambda(\tilde{Q}\setminus\tau^{n_i p}(A)) = 0$$

and $\tilde{Q} \subseteq A$ λ-a.e.

The set $\tilde{Q}$ is open λ-a.e., so $\tau^{-n}(\tau^n(\tilde{Q}))$ is also open λ-a.e., for any $n \geqslant 1$ (τ is a piecewise diffeomorphism). Thus, the set

$$W = \bigcup_{n\geqslant 0} \tau^{-n}(\tau^n(\tilde{Q}))$$

is open λ-a.e. Moreover, $W \in \mathfrak{A}_\infty(\tau)$, $W \subseteq A$ λ-a.e., and since A is an atom of $\mathfrak{A}_\infty(\tau)$, $W = A$ λ-a.e. Hence A is open λ-a.e. ∎

Corollary 1. Let τ satisfy the assumptions of Lemma 1 and let $\mathfrak{A}_\infty(\tau)$ be finite λ-a.e. Let f be the density of an ergodic acim. Then the support of f is an open set λ-a.e.

Proof. Immediate consequence of Theorem 2. ∎

Precisely as in ref. 6, we can prove:

Theorem 3. Let τ satisfy the assumptions of Lemma 1 and let $\mathfrak{A}_\infty(\tau)$ be finite λ-a.e. Let μ be an acim for τ. Then:

(a) If $\mu(\bigcup_{n\geqslant 0}\tau^n(U)) = \mu(S)$ for any open set $U \subset S$, then τ is ergodic.

(b) If $\mu(\bigcup_{n\geqslant 0}\tau^{kn}(U)) = \mu(S)$ for any open set $U \subset S$ and any $k \in N$, then τ is exact.

(c) If (τ, μ) is weakly mixing, then its natural extension is isomorphic to a Bernoulli shift.

We can now establish bounds on the number of ergodic τ-invariant absolutely continuous measures for the higher-dimensional Jabłonski transformations.

Let $S = I^N = [0, 1]^N$. We assume that any $S_i \in \mathscr{P}$, $i = 1,..., m$, is a rectangle. A transformation τ: $I^n \to I^n$ is called a Jabłonski transformation if it is piecewise C^2 and expanding, given by the formulas

$$\tau(x_1,..., x_N) = (\varphi_{1i}(x_1),..., \varphi_{ni}(x_n))$$

for $(x_1,..., x_N) \in S_i$, $i = 1,..., m$.

Definition 1. We say that a point x belongs to a set X with respect to the measure λ, and we write $x \in X$ (w.r.t. λ) if and only if there exists $\varepsilon > 0$ such that $\lambda(B(x, \varepsilon) \cap X) = \lambda(B(x, \varepsilon))$, where $B(x, \varepsilon)$ is the ball with center x and radius ε.

Definition 2. We define a function η: for any $x \in S$

$$\eta(x) = \{\text{number of } S_i \text{ such that } x \in S_i\}$$

We then define the crossing points to be local strict maxima of the function η. The set of crossing points will be denoted by $\mathfrak{C}$.

For example, if τ is a Jabłonski transformation, then $\mathfrak{C}$ is the set of vertices of elements of $\mathscr{P}$ which lie in the interior of S.

Definition 3. For a Jabłonski transformation, we define a number M_N related to the geometry of the partition $\mathscr{P}$. For any fixed $z \in \mathbb{R}$, let $H_{N-1}^{(j)}(z)$ denote the $(N-1)$-dimensional hyperplane given by the equation $x_j = z$, $j = 1, 2,..., N$. Let

$$M_N = \max_{z \in \mathbb{R}} \max_{1 \leqslant j \leqslant N} \{\text{number of } S_i \text{ such that } H_{N-1}^{(j)}(z) \cap \operatorname{Int}(S_i) \neq \varnothing\}$$

Theorem 4. Let $\tau: I^N \to I^N$ be a Jabłonski transformation and let

$$\Lambda = \inf\{J(\tau, x): x \in S\}$$

If $\Lambda/M_N > 1$, then the number of ergodic, absolutely continuous τ-invariant measures is at most equal to the number of crossing point, $\#\mathfrak{C}$.

Proof. Let μ be an ergodic measure of τ. By Lemma 2 we can find an open set B equal λ-a.e. to the support of μ. Let A_0 be a rectangular non-empty subset of B lying completely in one of the domains S_i, $i = 1,..., m$. We proceed inductively: if A_n is already defined, we define A_{n+1} to be the intersection of $\tau(A_n)$ with sets S_i, $i = 1,..., m$, which has maximal λ measure.

Then, if $\tau(A_0)$, $\tau(A_1)$,..., $\tau(A_n)$ do not include a crossing point in their interiors, we have

$$\lambda(A_{n+1}) \geqslant \frac{\Lambda}{M_N} \lambda(A_n) \geqslant \left(\frac{\Lambda}{M_N}\right)^{n+1} \lambda(A_0)$$

Since $\Lambda/M_N > 1$, we have $\lambda(A_{n+1}) \to \infty$, as $n \to \infty$, which is impossible. Thus some image $\tau^k(A_0)$ contains a crossing point in its interior, which implies that this crossing point belongs to the supp μ w.r.t. λ. Since the supports of ergodic acim's are mutually disjoint λ-a.e., the conclusion of the theorem follows. ∎

Note that $M_1 = 1$. Hence, in dimension 1, $\inf|\tau'| > 1$ implies that the number of ergodic acim's is at most the number of discontinuity points. This is a well-known result.[1]

The following two corollaries may sometimes be used to improve the estimate of Theorem 4.

Definition 4. For a Jabłonski transformation, we define a number K_N related not only to the geometry of the partition $\mathscr{P}$, but also to the structure of the transformation τ. Let

$$K_N = \max_{z \in \mathbb{R}} \max_{1 \leqslant j \leqslant N} \max_{1 \leqslant k \leqslant m} \{\text{number of } S_i \text{ such that } H^{(j)}_{N-1}(z) \cap \tau(S_k) \cap \operatorname{Int}(S_i) \neq \varnothing\}$$

Corollary 2. If $\Lambda/K_N > 1$, then the number of ergodic, absolutely continuous τ-invariant measures is at most equal to the number of crossing points, $\#\mathfrak{C}$.

Proof. The same as that of Theorem 4. ∎

Remark. Since $\Lambda > \sigma^{-N}$, it is enough to assume $\sigma^N K_N < 1$.

Remark. In the proofs of Theorem 4 and Corollary 2, we only used the fact that the elements of the partition $\mathscr{P}$ are rectangles and that the τ-image of a rectangle is again a rectangle (with faces parallel to the coordinate hyperplanes.) With this in mind we can generalize Theorem 4 even further.

Definition 5. Let V be any open convex subset of S. We define a number C_N as follows:

$$C_N = \max_{V} \max_{1 \leqslant j \leqslant N} \max_{1 \leqslant k \leqslant m} \{\text{number of } S_i \text{ such that } V \cap \tau(S_k) \cap \operatorname{Int}(S_i) \neq \varnothing\}$$

Corollary 3. Assume that the elements of $\mathscr{P}$ are convex and the transformation τ maps convex sets into convex sets. If $\Lambda/C_N > 1$, then the number of ergodic, absolutely continuous τ-invariant measures is at most equal to the number of crossing points, $\#\mathfrak{C}$.

Proof. The same as that of Theorem 4. ∎

Note that S itself does not have to be convex, but since each of the elements of $\mathscr{P}$ is convex, the inner boundaries of these elements must be straight lines.

Definition 6. The crossing points c_1, $c_2 \in \mathfrak{C}$ belong to the same dependent class if, for any $\varepsilon > 0$,

$$\lambda(\tau^k(B(c_1, \varepsilon)) \cap \tau^l(B(c_2, \varepsilon))) > 0$$

where k, $l \geqslant 0$.

Corollary 4. Under the assumptions of Theorem 4, Corollary 2, or Corollary 3, the number of ergodic, absolutely continuous τ-invariant measures is at most equal to the minimal number of dependent classes into which the crossing points can be divided.

Proof. Let c_1, c_2 be in the same dependent class. We can assume that there are ergodic acim's μ_1, μ_2 such that $c_1 \in \operatorname{supp} \mu_1$ w.r.t. λ and $c_2 \in \operatorname{supp} \mu_2$ w.r.t. λ. (Otherwise, at least one of these points is unimportant for the estimate of the number of ergodic acim's given in Theorem 4.). We can choose $\varepsilon > 0$ to satisfy

$$B(c_1, \varepsilon) \subset \operatorname{supp} \mu_1, \quad \lambda\text{-a.e.}$$

and

$$B(c_2, \varepsilon) \subset \operatorname{supp} \mu_2, \quad \lambda\text{-a.e.}$$

The dependence of c_1 and c_2 implies

$$\lambda(\tau^k(\operatorname{supp} \mu_1) \cap \tau^l(\operatorname{supp} \mu_2)) > 0$$

for some k, $l \geqslant 0$, so $\mu_1 = \mu_2$. ∎

Below we give another method of estimating the number of ergodic acim's. The result may be applied in a general situation, but does not give an explicit bound on the number of ergodic acim's.

Let $\partial_n(\mathscr{P}) = \bigcup_{i=1}^m \partial(\tau^{-n}(S_i))$, $n = 0, 1,...,$ and let $P^* = \bigcup_{n>0} \bigcup_{i=1}^m \partial(\tau^n(S_i))$ be the union of all the boundaries of the forward images of elements of the defining partition.

Theorem 5. Let $\tau: S \to S$ satisfy the conclusions of Theorem 1, where f is the density of an ergodic measure μ. Let A be any set such that $\bigcup_{n=0}^{\infty} \tau^{-n}(A)$ is dense in S and $A \cap P^* = \emptyset$. Then there exists an $a \in A$ such that $a \in \operatorname{supp}(f)$ (w.r.t. λ) and the number of ergodic acim's is at most the cardinality of A.

Remark. Theorem 4 and Corollary 2 state that the preimages of crossing points are dense in S.

Proof. By Theorem 2, there exists an open set B such that $B = \operatorname{supp}(f)$ λ-a.e. Since $\bigcup_{n \geq 0} \tau^{-n}(A)$ is dense in S, there exists an $a \in A$ and a smallest integer k such that $\tau^{-k}(a) \in B$. Since $a \notin P^*$, $\tau^{-k}(a) \notin \partial_0(\mathscr{P}) \cup \partial_1(\mathscr{P}) \cup \cdots \cup \partial_{k-1}(\mathscr{P})$, and thus there exists $\delta > 0$ such that $B(\tau^{-k}(a), \delta) \subset B$ and $\tau^k_{|B(\tau^{-k}(a), \delta)}$ is a diffeomorphism. Thus $\tau^k(B(\tau^{-k}(a), \delta)$ is an open neighborhood of a and $\tau^k(B(\tau^{-k}(a), \delta)) \subseteq \operatorname{supp}(f)$ λ-a.e., i.e., $a \in \operatorname{supp}(f)$ (w.r.t. λ). Since the supports of different ergodic acim's are disjoint λ-a.e., the last conclusion of the theorem follows. ∎

Remarks. (1) Given that the support of the density f of an acim is an open set λ-a.e., does this shed any light on whether f is bounded above and/or below on its support?

(2) Random higher-dimensional point transformations can be treated in a way that is analogous to ref. 21. For example, we can show that the minimum of all the upper bounds for the number of ergodic acim's for the component transformations is an upper bound for the number of ergodic acim's for the random map.

3. EXAMPLES OF JABŁONSKI TRANSFORMATIONS

(i) Let $S = [-2, 2] \times [-3, 3]$ and $\mathscr{P} = \{S_1, S_2, S_3, S_4\}$, where S_i is the intersection of S with the ith quadrant of the plane (Fig. 1). Define $\tau_1 = \tau_{|S_1}$ by $(x, y) \to (-4/3y + 2, 3/2x)$, i.e., τ_1 maps S_1 linearly onto $S_1 \cup S_2$. Let τ_2 map S_2 onto $S_1 \cup S_2$ in the same way, and let τ_3, τ_4 map S_3 and S_4, respectively, onto $S_3 \cup S_4$ in the same manner. Then $\Lambda = K_2 = 2$. It is obvious that τ admits two ergodic acim's. This shows that the condition $\Lambda/K_N > 1$ is necessary in Corollary 2 and also shows that in general the number of crossing points $\#\mathfrak{C}$ is not a bound for the number of ergodic acim's.

(ii) Let $S = [-1, 1] \times [-1, 1]$ and $\mathscr{P} = \{S_1, S_2, S_3, S_4\}$, where S_i is the intersection of S with the ith quadrant of the plane. We define $\tau_i = \tau_{|S_i}$ by

$$\tau_i(x, y) = (ax - \operatorname{sgn}(x)\, b, ay - \operatorname{sgn}(y)\, b)$$

If $1<a<2$ and $(a-1)<b<1$, then τ is a piecewise expanding Jabłonski transformation and $\tau(S)\subset S$. For values in small intervals around $a=1.18$ and $b=0.85$, using the computer we obtained strong evidence[12] that τ has four ergodic acim's. The computer images of the supports of these measures are shown on Figs. 2–5. This gives evidence that $\#\mathscr{P}-1$ (which might be a resonable generalization of the one-dimensional result) is not, in general, a bound for number of acim's.

(iii) In ref. 16, we model an N-site cellular automaton by an N-dimensional point transformation. Let L be a lattice of cells having arbitrary shape and dimension. The values of the N cells are described by N real variables $(x_1,\ldots, x_N)\in I^N$, $I=[0, 1]$. The evolution of values of the N-cell system is described by the point transformation τ. To define τ, we consider a partition of I^N into 2^N subsets:

$$I_{i_1 i_2\cdots i_N}=I_{i_1}\times I_{i_2}\times\cdots\times I_{i_N}$$

where each I_i is one of the two intervals which partition I, namely $[0, 1/2)$ or $[1/2, 1]$. To define τ, it suffices to define the ith component $\tau^{(i)}$. Fix $N-1$ indices: $j_1,\ldots, j_{i-1}, j_{i+1}, j_{i+2},\ldots, j_N$, each taking a value in the set

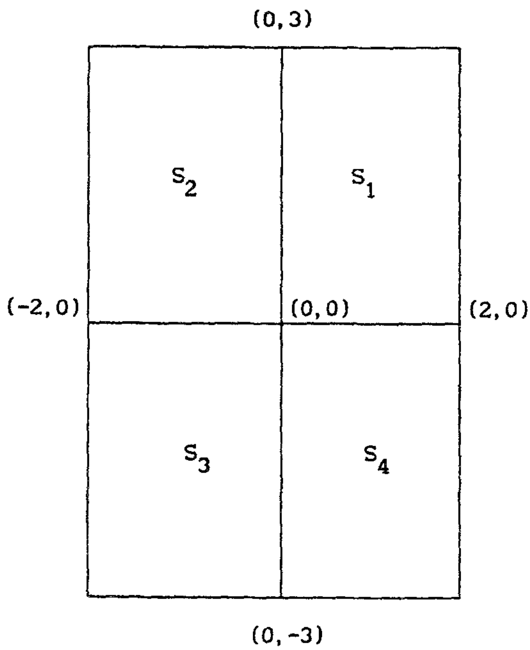

Fig. 1

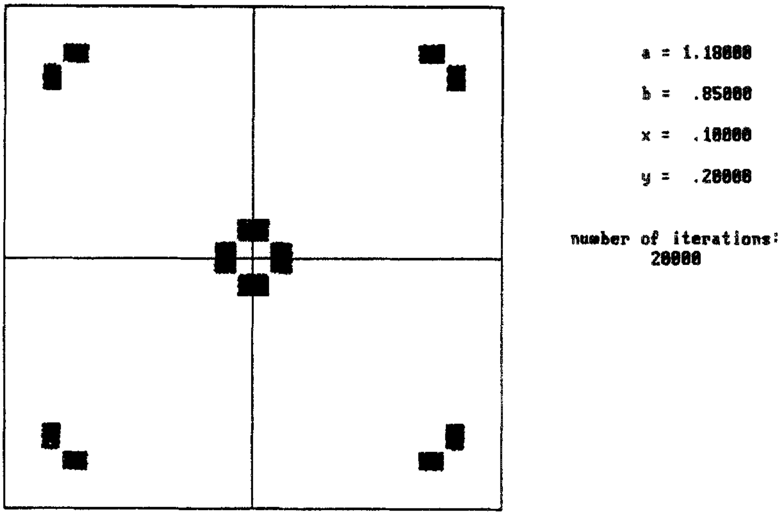

Fig. 2

$\{0, 1\}$, where 0 represents the interval $[0, 1/2)$ and 1 the interval $[1/2, 1]$. On the set

$$I_{j_1} \times I_{j_2} \times \cdots \times I_{j_{i-1}} \times [0, 1] \times I_{i_{i+1}} \times I_{j_{i+1}} \times \cdots \times I_{j_N}$$

$\tau^{(i)}$ depends only on x_i and for any fixed

$$x_1 \in I_{J_1}, x_2 \in I_{j_2}, \ldots, x_{i-1} \in I_{j_{i-1}}, x_{i+1} \in I_{j_{i+1}}, \ldots, x_N \in I_N$$

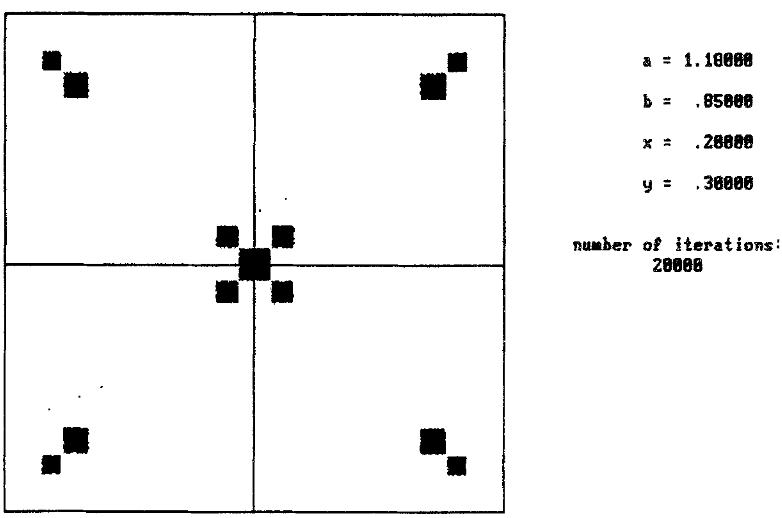

Fig. 3

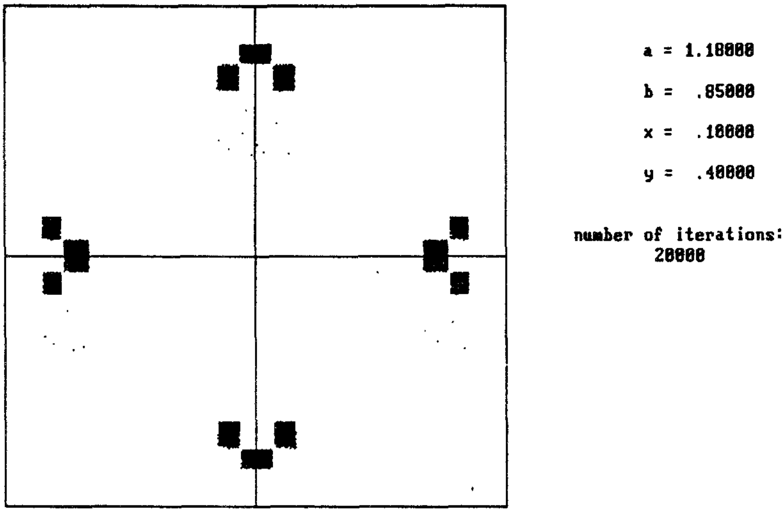

Fig. 4

is a one-dimensional, piecewise smooth, expanding transformation, as shown in Fig. 6.

In order to use the bound of Corollary 2, we need $\Lambda > 2^{N-1}$. Since $\Lambda > \sigma^{-N}$, we require the transformation to satisfy $\sigma < 2^{1/N}/2$. Since there is only one crossing point, τ has a unique acim.

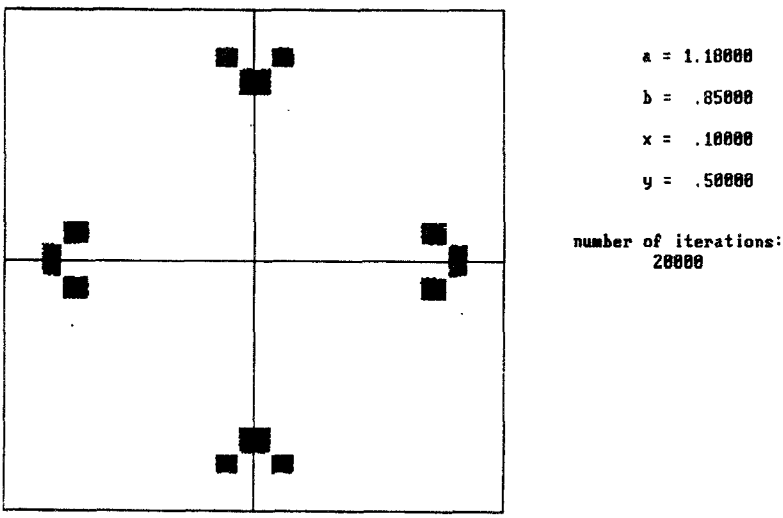

Fig. 5

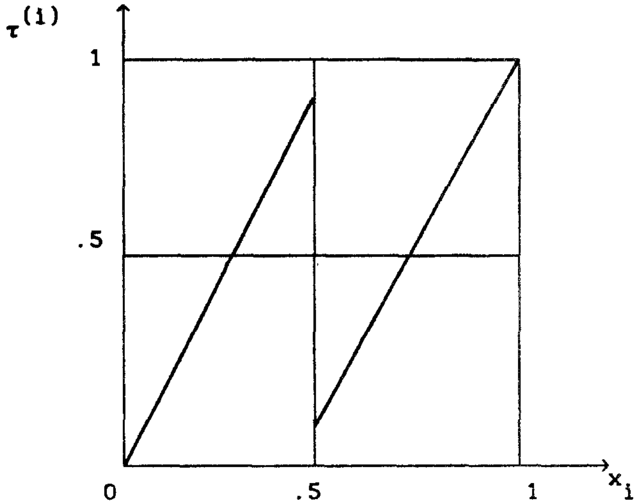

Fig. 6

4. EXAMPLE OF A GENERAL TRANSFORMATION

In this section we will construct two-dimensional piecewise expanding C^2 transformations on a fixed finite partition, but which have an arbitrarily large number of ergodic, absolutely continuous invariant measures.

Let S be a bounded closed region in $\mathbb{R}^2$ with subregions $S_1, S_2, \ldots, S_m$. Let S_1 and S_2 share a boundary segment which is a line L (Fig. 7). Let

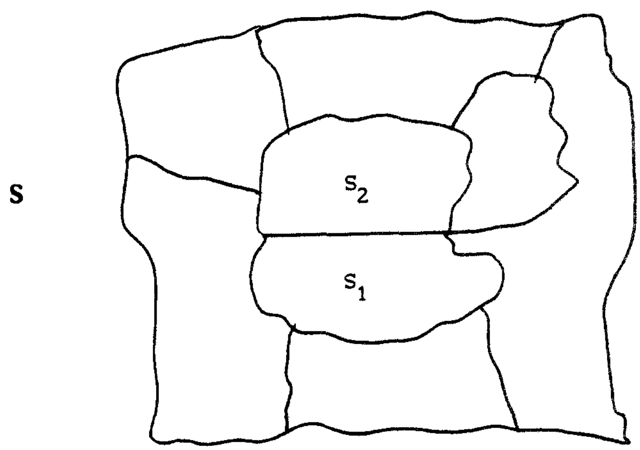

Fig. 7

$M=2k+1$ be an arbitrary odd positive integer. Let l be the length of L and let us mark off points p_i at distances $il/(M+1)$, $1\leqslant i\leqslant M$, from an endpoint of L. We construct a line segment centered at p_i (in L) of length

$$\delta\leqslant\frac{1}{M+1}\frac{1}{p}$$

where $p\geqslant 3$ will be chosen later. We now construct isosceles right triangles $T_i^{(j)}$, $1\leqslant i\leqslant M$, $j=1, 2$, of sides δ, δ, $2^{1/2}\delta$, as shown in Fig. 8.

To define $\tau_1=\tau_{|S_1}$, we proceed as follows:

Step 1: Reflect S_1 about the perpendicular line to L at p_{k+1} (Fig. 9). This maps the base of $T_i^{(1)}$ to the base of $T_{M+1-i}^{(1)}$ and flips the triangles. Note that for $i=k+1$ the base maps to itself, i.e., is invariant.

Step 2: Let C_i be vertices of the reflected T_i at the right angle. Perform a homothetic dilation by $\sqrt{2}$ at C_i followed by a 45° rotation as shown in Fig. 10. Note that C_iB_i' is the image of the base of triangle $T_{M+1-i}^{(1)}$.

Step 3: Extend τ_1 defined above on triangles $T_i^{(1)}$ to a neighborhood of L in S_1. That is, we define τ_1 between the two adjacent triangles so that τ_1 is expanding. This can be done as suggested in Fig. 11, provided the spacing between adjacent triangles is sufficiently large in relation to δ, i.e.,

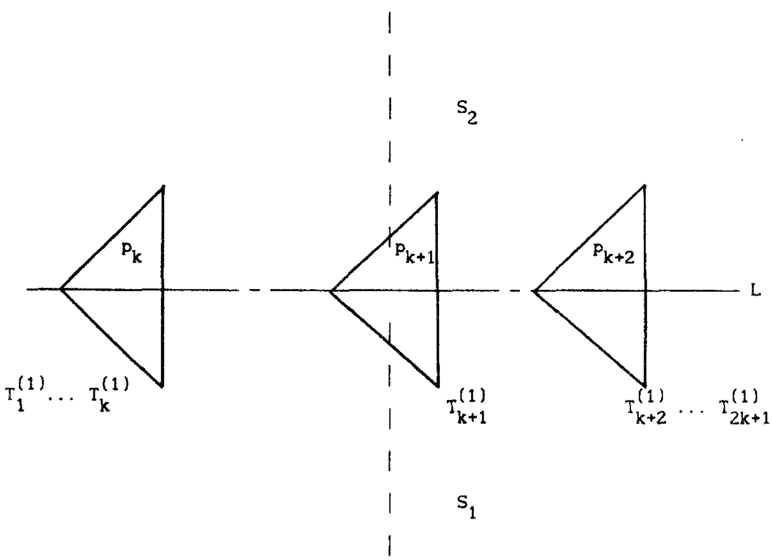

Fig. 8

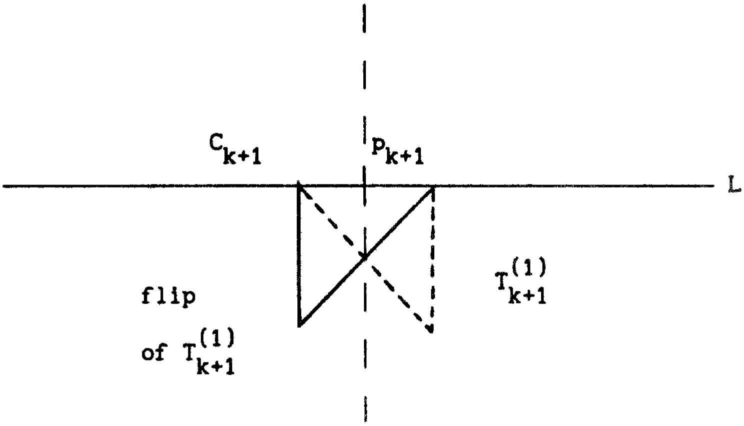

Fig. 9

provided δ is sufficiently small. Choose p accordingly. This defines τ_1 within a distance δ of L.

Step 4: Extend τ_1 to the remainder of S_1 so that it is expanding and C^2 on S_1.

By symmetry, we may repeat the same construction for S_2, i.e., repeat the construction for S_1 symmetrically with respect to L to get τ_2 within a distance δ of L, and extend as in Step 4.

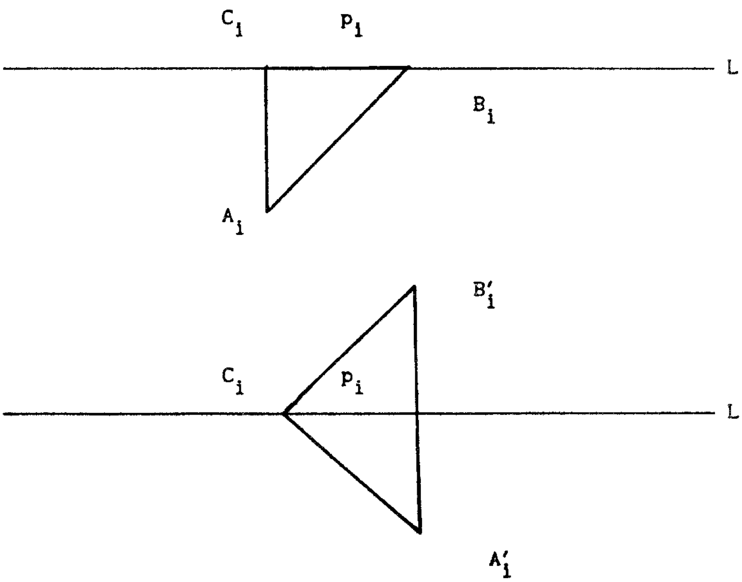

Fig. 10

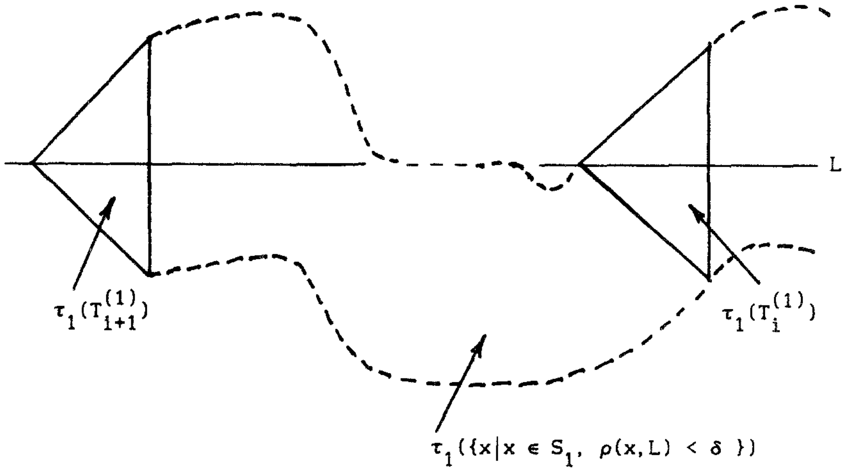

Fig. 11

Let

$$E_i = T_i^{(1)} \cup T_i^{(2)} \cup T^{(1)}_{M+1-i} \cup T^{(2)}_{M+1-i}, \qquad 1 \leqslant i \leqslant M$$

Define τ on the rest of S so that

$$\tau(S_j) \cap \bigcup_{i=1}^{M} E_i = \varnothing \qquad \text{for } j > 2$$

Then $\tau(E_i) = E_i = \tau^{-1}(E_i)$, for $1 \leqslant i \leqslant M$, so each E_i is an invariant set of positive Lebesgue measure, and hence supports an ergodic, absolutely continuous invariant measure. Since there are $k+1$ distinct E_i and k is arbitrary, we can have an arbitrarily large number of ergodic, absolutely continuous invariant measures.

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